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Journal of Computational Physics 191 (2003) 322–327
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Short Note

Including electron inertia without advancing electron flow

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Received 15 January 2003; received in revised form 8 May 2003; accepted 5 June 2003

1. Introduction

There are a variety of plasma physics problems where including electron inertia can be important, e.g., magnetic reconnection [1–9], inertial Alfvén waves for auroral electron acceleration [10,11] and electron-temperature-gradient turbulence [12,13]. There has also been recent interest in closure of fluid-electron equations for gyrokinetic turbulence simulations [14–19] where electron inertia is typically neglected, but could be important near mode rational surfaces [20]. Here, we describe a benign way to include the effects of electron inertia without advancing the electron momentum equation and thereby avoiding the associated $k_{\|} v_{\mathrm{te}} \Delta t$ Courant condition or subsequent implicit methods. The effect of electron inertia has been included in simpler Electron-MHD models [4,7,21,22] without advancing the electron momentum equation directly. We apply a related technique to drift-fluid electron models appropriate for gyrokinetic simulation of well-magnetized plasmas. First, we discuss our technique for including the electron inertia in drift-fluid electron equations. Next, we use a simple example of von Neumann stability analysis to demonstrate that the new scheme is stable whereas trivial finite-differencing of the electron flow is shown to be unstable. Finally, a similar method is discussed for extended-MHD models and shown to be a generalization of the Electron-MHD equations.

2. Electron inertia in a electron drift-fluid model

A numerical stability problem arises when a simple drift-fluid electron model is used to simulate Alfvén waves in a regime where the electron inertia is important. We have used a drift-fluid electron model with electron inertia for two distinct applications. First, we have modeled Alfvén waves in the terrestrial magnetosphere and ionosphere. The electron inertia term is necessary for the generation of parallel electric fields. Second, very similar equations are used in kinetic hybrid models for microturbulence [14–18]. Both the microturbulence and magnetospheric simulations include more complete gyrokinetic or gyrofluid models for the ions. However, the difficulty with the inertia term can be best demonstrated with the following simple one-dimensional electron equations along a magnetic field line.

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0021-9991/$ - see front matter
doi:10.1016/S0021-9991(03)00305-X

The ion polarization drift effect is retained by using the gyrokinetic quasi-neutrality condition [23]. We advance the drift-fluid electron continuity equation [24] and the gauge definition of electric field in time using

$$\frac{\partial n_e}{\partial t} = -B\nabla_{||}\left(\frac{n_e u_{||e}}{B}\right), \tag{1}$$

$$\frac{\partial A_{||}}{\partial t} = -E_{||} - \nabla_{||}\Phi, \tag{2}$$

where n_e is the electron density, $u_{||e}$ is the parallel electron flow, and $A_{||}$ is the vector potential parallel to **B**. Φ is solved for by using gyrokinetic Poisson equation (or quasi-neutrality condition)

$$\rho_i^2\nabla_\perp^2\frac{e\Phi}{T_i} = \frac{\delta n_e}{n_e} - \frac{\delta n_i}{n_i}, \tag{3}$$

where T_e is the electron temperature, $\rho_i^2 = m_i T_i/e^2B^2$ is the ion gyroradius, T_i is the ion temperature, and we have taken the small $k_\perp^2\rho_i^2$ limit for clarity in Eq. (3). $u_{||e}$ is evaluated by solving Ampere's law backwards

$$e\mu_0 n_e u_{||e} = \nabla^2 A_{||} + e\mu_0 n_i u_{||i}. \tag{4}$$

Finally $E_{||}$ is updated using the drift-fluid electron momentum equation in the form of a generalized Ohm's law

$$\frac{e}{m_e}n_e E_{||} = -\nabla_{||}\left(\frac{n_e T}{m_e}\right) - \frac{\partial\left(n_e u_{||e}\right)}{\partial t}. \tag{5}$$

We set δn_i and $u_{||i}$ to zero because the Alfvén wave can be demonstrated with electron equations alone. However, in our more complete simulation we include either full gyrofluid or gyrokinetic ion response, and the methods discussed here are easily generalized to include ions. Linearizing Eqs. (1)–(5) produces the following dispersion relation

$$\frac{\omega^2}{k_{||}^2} = v_a^2\frac{1+k_\perp^2\rho_s^2}{1+k_\perp^2\lambda_e^2},$$

which includes both the kinetic and inertial corrections to the Alfvén wave. $v_a^2 = B^2/\mu_0 m_i n_i$ is the Alfvén speed, $\rho_s^2 = m_i T_e/e^2B^2$, and $\lambda_e^2 = m_e/n_e\mu_0 e^2$ is the electron skin depth.

After we advance Eqs. (1)–(4), we can evaluate $\partial u_{||e}/\partial t$ in Eq. (5) using a backward finite-difference in time. This method has been shown to be stable in the regime where ion kinetic effects are more important than electron inertia [18]. However, this method caused numerical instability in our simulation when $\lambda_e > \rho_i$. By examining the contributions of the various terms in our simulation, we clearly identified that the term $(u_e^{n+1} - u_e^n)/\Delta t$ was the cause of the numerical instability. This motivated using the following method of evaluating $\partial u_{||e}/\partial t$. Rather than directly evaluating $\partial n_e u_{||e}/\partial t$, we use Eqs. (4) and (2) to obtain

$$\frac{\partial n_e u_{||e}}{\partial t} = \frac{1}{\mu_0 e}\nabla^2\frac{\partial A_{||}}{\partial t} = \frac{-1}{\mu_0 e}\nabla^2\left(E_{||} + \nabla_{||}\Phi\right). \tag{6}$$

We then substitute Eq. (6) into Eq. (5) to obtain the following generalized parallel Ohm's law:

$$\frac{e}{m_e}n_e\left(1+\lambda_e^2\nabla^2\right)E_{||} = -\nabla_{||}\left(\frac{n_e T}{m_e}\right) + \frac{1}{\mu_0 e}\nabla^2\left(\nabla_{||}\Phi\right), \tag{7}$$

which allows us to evaluate $E_{||}^{n+1}$ in a time centered manner. Eq. (7) has the form of a modified parallel Ohm's law (two new terms appearing). The last term on the right-hand-side is straight-forward to evaluate.

The operator $(1+\lambda_e^2\nabla^2)$ on the left-hand-side is trivial to invert spectrally, or could involve a narrowly banded matrix inversion more generally.

3. Stability analysis

In this section von Neumann stability analysis is used to show that explicit finite-differencing of the inertia term in the parallel Ohm's law, Eq. (5) is unstable. We then analyze the finite-difference equations using the alternative form of the parallel Ohm's law, Eq. (7), and obtain a Courant condition (conditional stability). To obtain an analytically tractable result, we use a first-order Lax–Wendroff method applied to Eqs. (1)–(5), and use a simple backward difference of $\partial u_{||e}/\partial t$ in Eq. (5). The Lax–Wendroff method [25] for a flux-conserving equation $\partial u/\partial t = \partial F(u)/\partial x$, is

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) + \frac{\Delta t}{2\Delta x}\left[F\left(u_{j+1}^n\right) - F\left(u_{j-1}^n\right)\right]. \tag{8}$$

We make the standard assumptions that all the quantities are periodic in space, and grow exponentially in time, hence, $n_j^n = n^n\,\mathrm{e}^{\mathrm{i}k_{||}j\Delta x}$, $E_j^n = E^n\,\mathrm{e}^{\mathrm{i}k_{||}j\Delta x}$, and so forth. Defining $\phi \equiv k_{||}\Delta x$ and $G \equiv E^{n+1}/E^n$, we can solve for G, the amplification factor, and study the stability of the scheme

$$G\cos^2\phi + G^3 - 2G^2\cos\phi + G\left(\frac{\Delta t}{\Delta x}\right)^2 v_a^2\sin^2\phi = \alpha(-G^2 + G + G\cos\phi - \cos\phi) - \psi\left(\frac{\Delta t}{\Delta x}\right)^2 G\sin^2\phi, \tag{9}$$

where $\alpha = \lambda_e^2 k_\perp^2$, and $\psi = v_a^2 + v_{te}^2\lambda_e^2 k_\perp^2$.

When $\phi = \pi$, $k_{||} = \pi/\Delta x$ which represents the Nyquist wavelength which is the smallest wavelength that can be resolved. This yields

$$G^2 + \alpha G + G - \alpha = 0. \tag{10}$$

A plot of $|G(\alpha)|$ is shown in Fig. 1. We see that for finite α there exists a growing solution for $|G|$. For small α we have $|G| \simeq 1 + 2\alpha$. Therefore, an absolute instability exists for the first-order Lax–Wendroff method when the inertia term in Eq. (5) is finite-differenced in time and $k_\perp$ is finite. Shown in Fig. 1 are the results of a first-order simulation varying α. This obviously does not automatically mean that all schemes will be unstable. However, we have numerically tested a second-order Lax–Wendroff scheme and observe that increasing $k_\perp$ causes numerical instability in the electron inertial regime as well (Fig. 1). For $\alpha \lesssim 1.3$ the second-order scheme is found to be stable. Solving for for G for the second-order scheme required solving a quintic equation for α and was not analytically tractable.

Next, we do the same stability analysis, but rather than finite-differencing the inertia term, the last term in Eq. (5), we use the new parallel Ohm's law, Eq. (7), obtaining

$$G^2 - 2G\sin\phi + \cos^2\phi - \sqrt{\frac{\psi}{1+\alpha}}\sin^2\phi = 0. \tag{11}$$

Solving this quadratic for G yields

$$G = \cos\phi \pm \mathrm{i}\sqrt{\frac{\psi}{1+\alpha}}\frac{\Delta t}{\Delta x}\sin\phi. \tag{12}$$

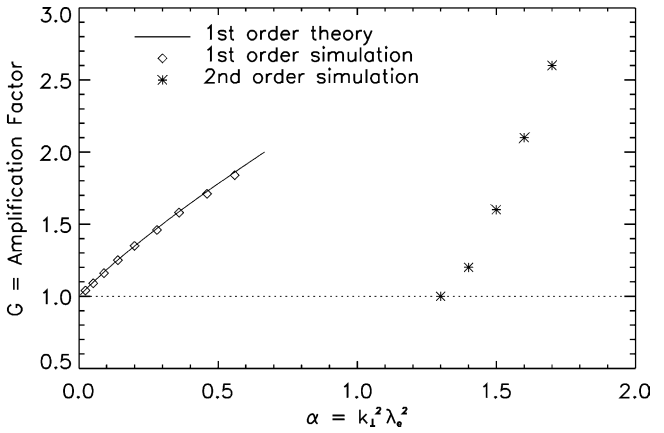

Fig. 1. α versus amplification factor $|G|$. $|G| > 1$ demonstrates absolute instability of finite α when electron inertia is explicitly finite differenced.

which shows the scheme is conditionally stable and constrained by the following Courant condition,

$$v_a \sqrt{\frac{1 + k_\perp^2 \rho_s^2}{1 + k_\perp^2 \lambda_e^2}} \frac{\Delta t}{\Delta x} \leqslant 1. \tag{13}$$

Therefore, using the time-centered substitution we describe here, i.e., using the modified Ohm's law, Eq. (7), conditional stability is achieved.

4. Application to MHD equations

In this section we apply a similar technique as discussed in Section 2 to include the electron inertia in the one-fluid generalized Ohm's law. The result is a generalization of the electron MHD equations [4,7,21,22]. We assume quasi-neutrality and begin with the usual MHD-Maxwell equations. **B** is obtained using Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{14}$$

and **E** is solved for using the generalized Ohm's law

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \frac{\eta}{\mu_0} \nabla \times \mathbf{B} + \frac{1}{\mu_0 e n} (\nabla \times \mathbf{B}) \times \mathbf{B} - \frac{1}{en} \nabla \cdot \mathbf{\Pi}_e - \frac{m_e}{ne} \frac{\partial (n \mathbf{u}_e)}{\partial t}, \tag{15}$$

where the displacement current in Maxwell's equations is neglected and Ampere's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ is used to eliminate **J** in the $\eta \mathbf{J}$ term and the Hall term. $\mathbf{\Pi}_e$ is the electron pressure tensor and is determined from the electron distribution function $\mathbf{\Pi}_e \equiv \int \mathbf{v}\mathbf{v} \delta f_e d^3 v$. $\mathbf{\Pi}_e$, the pressure term cannot be analytically eliminated like the inertia term, so we write it explicitly. For clarity, one could set $\mathbf{\Pi}_e = 0$ and assume the electron pressure term in the generalized Ohm's law is neglected. **u** is determined from the one-fluid MHD equation and the determination of $\mathbf{u}_e$ will be explained as follows.

We eliminate the time derivative in the electron inertia term, the last term in the generalized Ohm's law, Eq. (15), by using Faraday's law, Ampere's law and the ion momentum equation.

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial(\nabla \times \mathbf{B})}{\partial t} = \mu_0 e \left[\frac{\partial(n\mathbf{u}_e)}{\partial t} - \frac{\partial(n\mathbf{u}_i)}{\partial t} \right], \tag{16}$$

We can then rewrite the last term in Eq. (15) as

$$-\frac{m_e}{ne}\frac{\partial(n\mathbf{u}_e)}{\partial t} = \frac{m_e}{\mu_0 ne^2}\tilde{\nabla}^2\mathbf{E} - \frac{m_e}{ne}\frac{\partial(n\mathbf{u}_i)}{\partial t}, \tag{17}$$

where $\tilde{\nabla}^2 \equiv -\nabla \times (\nabla \times \mathbf{E})$. To eliminate $\partial(n\mathbf{u}_i)/\partial t$ we use the ion momentum equation

$$\frac{\partial(n\mathbf{u}_i)}{\partial t} = \frac{en}{m_i}(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \frac{1}{m_i}\nabla \cdot \mathbf{\Pi}_i, \tag{18}$$

where $\mathbf{\Pi}_i \equiv \int \mathbf{v}\mathbf{v}\delta f_i d^3v$. Substituting Eq. (18) into (17), we obtain

$$-\frac{m_e}{ne}\frac{\partial(n\mathbf{u}_e)}{\partial t} = \frac{m_e}{\mu_0 ne^2}\tilde{\nabla}^2\mathbf{E} - \frac{m_e}{m_i}(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + \frac{m_e}{m_i ne}\nabla \cdot \mathbf{\Pi}_i. \tag{19}$$

Using the fact that $m_e/m_i \ll 1$ we can neglect the last two terms on the right-hand-side of Eq. (19). The generalized Ohm's law, Eq. (15), then becomes

$$\left[1 - \lambda_e^2\tilde{\nabla}^2\right]\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \frac{\eta}{\mu_0}\nabla \times \mathbf{B} + \frac{1}{\mu_0 en}(\nabla \times \mathbf{B}) \times \mathbf{B} - \frac{1}{en}\nabla \cdot \mathbf{\Pi}_e. \tag{20}$$

E can be solved for by inverting the the operator on the left-hand-side of Eq. (20) using Fourier series or matrix inversion.

This new form of the generalized Ohm's law, Eq. (20), along with Faraday's law, Eq. (14) can be written in a form similar to the electron MHD equations [4,7,21,22]. Using the fact that $\nabla \times (\tilde{\nabla}^2 E) = \tilde{\nabla}^2(\nabla \times E)$, we can rewrite Eqs. (14) and (20) as

$$\frac{\partial \mathbf{B}'}{\partial t} = -\nabla \times \mathbf{E}', \tag{21}$$

where $\mathbf{B}' = [1 - \lambda_e^2\tilde{\nabla}^2]\mathbf{B}$ and $\mathbf{E}' = \left[1 - \lambda_e^2\tilde{\nabla}^2\right]\mathbf{E}$. These equations are similar in form to those used by Shay et al., see Eqs. (2)–(4) in [8]. To use Eq. (20) in extended-MHD computation, **u** would be determined from the one-fluid MHD equation and a pressure closure would be required to evaluate $\mathbf{\Pi}_e$.

Acknowledgements

We are grateful for support by NASA NAG5-11924 and DOE Grant DE-FCC02-02ER54669. Thanks to Dr. Y. Chen and B.I. Cohen for helpful discussions.

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